No. 2002-59

# ON STRATEGY AND RELATIVE SKILL IN POKER

By Marcel Dreef, Peter Borm, Ben van der Genugten

July 2002

ISSN 0924-7815

Discussion paper

# On Strategy and Relative Skill in Poker

Marcel Dreef[1,2], Peter Borm[1] and Ben van der Genugten[1]

June 26, 2002

**Abstract**

This article presents a generalization of the equilibrium analysis for the simple two-player poker game with alternate bidding of Von Neumann and Morgenstern. It approximates optimal play for this game if it is played with a regular deck of 52 cards and it discusses some strategic insights. In addition, the paper studies the relative skill level of this game.

**Keywords:** poker; optimal play; skill.
**JEL code:** C72.

## 1 Introduction

An interesting aspect of a game with chance elements is the skill that players can utilize to affect the game's outcome. This is obviously an interesting topic for discussion among players or between players and spectators, but besides that it is also important from a juridical point of view. In many countries, for instance in the Netherlands, the exploitation of casino games is regulated by a law that makes a distinction between *games of chance* and *games of skill*. As the name suggests, this distinction is based on the skill that can be used in the game. Games of skill can be freely exploited, whereas for the exploitation of games of chance a licence is required. The Dutch government has only granted such a licence to its own Holland Casino foundation.

Following the Dutch gaming act, Borm and Van der Genugten (2001) roughly defined the *skill level* of a game as the extent to which players can influence the game outcome, relative to the effect of the random device on this outcome. They developed a measure which enables an ordering of games on the real line between zero and one. A pure game of chance is assigned a skill level of zero, while a skill level of one corresponds to pure games of skill. A paper on skill in a somewhat broader context is Larkey, Kadane, Austin and Zamir (1997). That paper does not consider juridical problems, but it provides an interesting discussion on the interpretation and relevance of the concept of skill in analyzing and solving games.

A class of games that is interesting for application of the skill analysis is formed by poker games. So far, the skill computations in this area were restricted to simple examples that were

[1]CentER and Department of Econometrics and Operations Research, Tilburg University.
[2]Corresponding author. P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: dreef@uvt.nl.

used mainly for decorative purposes. Only games with very few cards were considered, while in a real poker game a player receives one hand from a range of 2,598,960 possible hands. An increase in the number of cards in the model will therefore be a good step towards a better approximation of the skill involved in real life poker games.

This paper approximates the situation with 2,598,960 hands by studying an extension of the two-player poker model with alternate bidding that was introduced by Von Neumann and Morgenstern (1944, chapter 19). This game is played as follows. First, both players pay an ante and receive a hand. Next, player 1 chooses between *betting* a fixed bet size, and *passing.* When player 1 has decided to bet, then player 2 can choose between *folding* and *calling* at the cost of the bet size. In the first case, he gives up the ante, while in the second case the same thing happens as when player 1 has passed: a showdown follows. In the showdown, the player with the better hand wins the pot. A specific variant of this game is also studied in the book of Binmore (1992).

In the model of Von Neumann and Morgenstern (1944) the hands of the players are drawn from a continuous uniform distribution on $[0, 1]$. This paper extends the model by considering general hand distributions. We compute the value of the game as well as optimal strategies for both players. Next, we translate our general strategic results to the situation where the game is played with a deck of 52 cards. We need this information to perform our final step, the approximation of the skill level of this game using the method described by Borm and Van der Genugten (2001). More details and applications of this skill analysis can be found in the book on skillful gambling by Van der Genugten, Das and Borm (2001).

The paper is organized as follows. First, we will give an exact description of the specific poker game under consideration in section 2. In section 3 we will compute the optimal strategies for both players and discuss equilibrium play in some more detail. Subsequently, we will approximate optimal play for the case where this poker game is played with a regular deck of 52 cards. This is the subject of section 4. Finally, we will measure the presence of skill in this variant of poker and present the results in section 5.

## 2 Game description

We will give a formal description of the rules of our poker game, to which we will refer as *minipoker* throughout this text. To begin the game, both players add an *ante* of size $a$ to the stakes. Then the cards are dealt. Instead of considering the $\binom{52}{5}\binom{47}{5}$ possible hand combinations that can be dealt in a general poker game, the hands are assumed to be real numbers, drawn from the interval $[0, 1]$. Player 1's hand is the value $u$ of a continuous random variable $U$ and player 2's hand is the value $v$ of a continuous random variable $V$. $U$ and $V$ are independently, identically distributed on $[0, 1]$ according to the cumulative distribution function $F : [0, 1] \to \mathbb{R}_+$. The function $f : [0, 1] \to \mathbb{R}_{++}$ denotes the probability density function for this distribution and is assumed to be *positive* and *continuous* on its domain.

After seeing his hand, player 1 can choose between *passing* and *betting*. If he passes, a *showdown* follows immediately. In the showdown, the players compare their hands and the player with the highest hand wins the pot. Betting means adding an extra amount $b$ to the stakes. After a bet by player 1, player 2 can decide to *fold* or to *call*. If he folds, then he loses his ante of $a$ to player 1. To call, player 2 must put an extra amount $b$ in the pot. In that case, a showdown follows and the player with the better hand takes the pot.

The difference with the case of Von Neumann and Morgenstern (1944) is that they only consider hands $u$ and $v$ that are drawn independently from uniform distributions on $[0, 1]$. Furthermore, they use a somewhat different terminology for the strategic options of the players. Whereas we distinguish the terms *betting* en *passing* for player 1 and *calling* and *folding* for player 2, Von Neumann and Morgenstern (1944) speak of *bidding high* and *bidding low* for both players.

Figure 1 displays our poker model in extensive form. Two possible hands, $u_1$ and $u_2$, for player 1 are shown. To keep the picture clear, player 2 is shown receiving his hand $v$ after player 1 has decided how to bet. Notice that the hand $v$ shown in the picture has a value that satisfies $u_1 < v < u_2$.

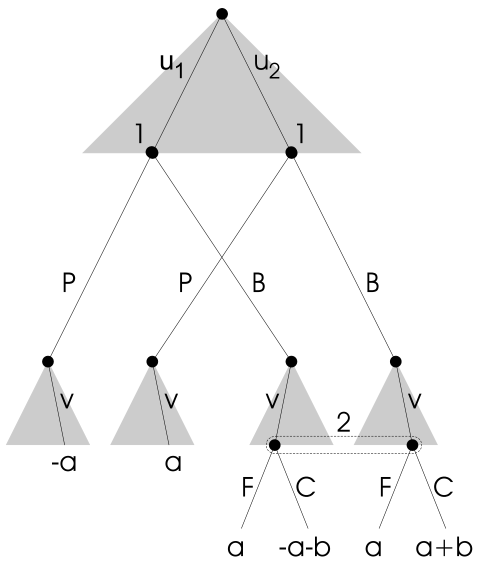


Figure 1: The extensive form of two-person minipoker ($u_1 < v < u_2$).

## 3 Optimal strategies

In this section we will search for the Nash equilibria of minipoker. We restrict attention to behavioural strategies that are measurable functions of the player's hands. The structure of the analysis is similar to the way Binmore (1992, chapter 12) explains a specific variant of this game.

Pure strategies for Von Neumann's poker model are functions $g : [0,1] \to \{P, R\}$ and $h : [0,1] \to \{F, C\}$. A mixed strategy is therefore something rather complex. Since the game has the property of perfect recall, according to Kuhn (1953) we can work with behavioural strategies as well, without really restricting the players in their possibilities.

A behavioural strategy for player 1 is a function $p : [0,1] \to [0,1]$, where $p(u)$ is the probability with which he bets if the value of his hand is $u$. Similarly, a behavioural strategy for player 2 is a function $q : [0,1] \to [0,1]$, where $q(v)$ is the probability with which he plans to call if he is dealt a hand with value $v$.

Suppose that the players use the behavioural strategies $p$ and $q$. Then, given a deal $(u, v)$, we can compute the expected gain $z(u, v)$ of player 1. This value depends on who has the better hand. If $u > v$,

$$\begin{aligned} z(u,v) &= \overbrace{ap(u)(1-q(v))}^{B,F} + \overbrace{(a+b)p(u)q(v)}^{B,C} + \overbrace{a(1-p(u))}^{P} \\ &= a + bp(u)q(v). \end{aligned}$$

If $u < v$,

$$\begin{aligned} z(u,v) &= \overbrace{ap(u)(1-q(v))}^{B,F} - \overbrace{(a+b)p(u)q(v)}^{B,C} - \overbrace{a(1-p(u))}^{P} \\ &= 2ap(u) - (2a+b)p(u)q(v) - a. \end{aligned}$$

Even though player 1 does not know what player 2 is holding, he can now compute the expectation with respect to $v$ of his own payoff for a given hand $u$.

$$\begin{aligned} E_1(u) &= \int_{v<u} z(u,v)f(v)dv + \int_{v>u} z(u,v)f(v)dv \\ &= \int_0^u (a + bp(u)q(v))f(v)dv + \int_u^1 (2ap(u) - (2a+b)p(u)q(v) - a)f(v)dv \end{aligned}$$

We can write

$$E_1(u) = p(u)S_1(u) + T_1(u),$$

where

$$\begin{aligned} S_1(u) &= 2a(1-F(u)) + b\int_0^u q(v)f(v)dv - (2a+b)\int_u^1 q(v)f(v)dv \qquad \text{and} \\ T_1(u) &= 2aF(u) - a. \end{aligned}$$

Analogously, for player 2, we get

$$\begin{aligned} E_2(v) &= -\int_{u<v} z(u,v)f(u)du - \int_{u>v} z(u,v)f(u)du \\ &= -\int_0^v (2ap(u) - (2a+b)p(u)q(v) - a)f(u)du - \int_v^1 (a + bp(u)q(v))f(u)du. \end{aligned}$$

Thus we can write

$$E_2(v) = q(v)S_2(v) + T_2(v),$$

where

$$\begin{aligned} S_2(v) &= (2a+b)\int_0^v p(u)f(u)du - b\int_v^1 p(u)f(u)du \qquad \text{and} \\ T_2(v) &= 2aF(v) - 2a\int_0^v p(u)f(u)du. \end{aligned}$$

When we look for a Nash equilibrium $(\widetilde{p}, \widetilde{q})$, all that matters are the signs of the functions $\widetilde{S}_1$ and $\widetilde{S}_2$, obtained by writing $q(v) = \widetilde{q}(v)$ and $p(u) = \widetilde{p}(u)$. How can we see this? Suppose that player 2 uses strategy $\widetilde{q}$. Then player 1 will get a payoff of $p(u)\widetilde{S}_1(u) + \widetilde{T}_1(u)$ if he raises with probability $p(u)$ when dealt $u$. If $\widetilde{S}_1(u) > 0$, the choice $p(u) = 1$ is optimal. If $\widetilde{S}_1(u) < 0$, the choice $p(u) = 0$ is optimal. Only if $\widetilde{S}_1(u) = 0$, other choices of $p(u)$ are optimal too. Applying similar considerations to player 2, we obtain the following criteria for equilibrium strategies $\widetilde{p}$ and $\widetilde{q}$:

$$\begin{aligned} \widetilde{S}_1(u) > 0 &\Rightarrow \widetilde{p}(u) = 1; \\ \widetilde{S}_1(u) < 0 &\Rightarrow \widetilde{p}(u) = 0; \\ 0 < \widetilde{p}(u) < 1 &\Rightarrow \widetilde{S}_1(u) = 0; \\ \widetilde{S}_2(v) > 0 &\Rightarrow \widetilde{q}(v) = 1; \\ \widetilde{S}_2(v) < 0 &\Rightarrow \widetilde{q}(v) = 0; \\ 0 < \widetilde{q}(v) < 1 &\Rightarrow \widetilde{S}_2(v) = 0. \end{aligned}$$

Figure 2(a) shows what the graph of $\widetilde{S}_2(v)$ looks like. Here and in other figures, for $F$ we have chosen the uniform distribution on the interval $[0, 1]$, while the ratio $\frac{b}{a}$ of the bet size and the ante is equal to 1. To check this, take a look at the expression for $\widetilde{S}_2(v)$. Since both $a$ and $b$ are positive numbers, $f(u)$ is assumed to be positive for all $u \in [0, 1]$ and $p(u)$ can only take nonnegative values, it follows that the function $\widetilde{S}_2$ is weakly increasing in $v$. Substituting $v = 0$ in the formula for $\widetilde{S}_2(v)$ yields $\widetilde{S}_2(0) \leq 0$, while substitution of $v = 1$ tells us $\widetilde{S}_2(1) \geq 0$. Since $\widetilde{S}_2$ is necessarily continuous, there exist numbers $x$ and $y$ such that $x$ is the smallest number in $[0, 1]$ for which $\widetilde{S}_2(x) = 0$ and $y$ is the largest number in $[0, 1]$ for which $\widetilde{S}_2(y) = 0$. Note that, unless $x = y$, the function $\widetilde{S}_2$ cannot be strictly increasing. The information about $\widetilde{S}_2$, summarized in figure 2(a), tells us much about the function $\widetilde{q}$. What we know about $\widetilde{q}$ is summarized in figure 2(c).

The expression for $\widetilde{S}_2(v)$ is informative about the function $\widetilde{p}$ too. Since $\widetilde{S}_2(v)$ is constant for $v$ on the interval $[x, y]$, we must have $\widetilde{p}(v) = 0$ on the interval $(x, y)$. However, $\widetilde{p}(v)$ cannot be zero on a larger open interval $I$, because this would imply that $\widetilde{S}_2(v)$ would then be constant on $I$. This constant would need to be zero, because $\widetilde{S}_2(v) = 0$ on the interval $[x, y]$. However, this contradicts the fact that $[x, y]$ is the largest interval on which $\widetilde{S}_2(v) = 0$.

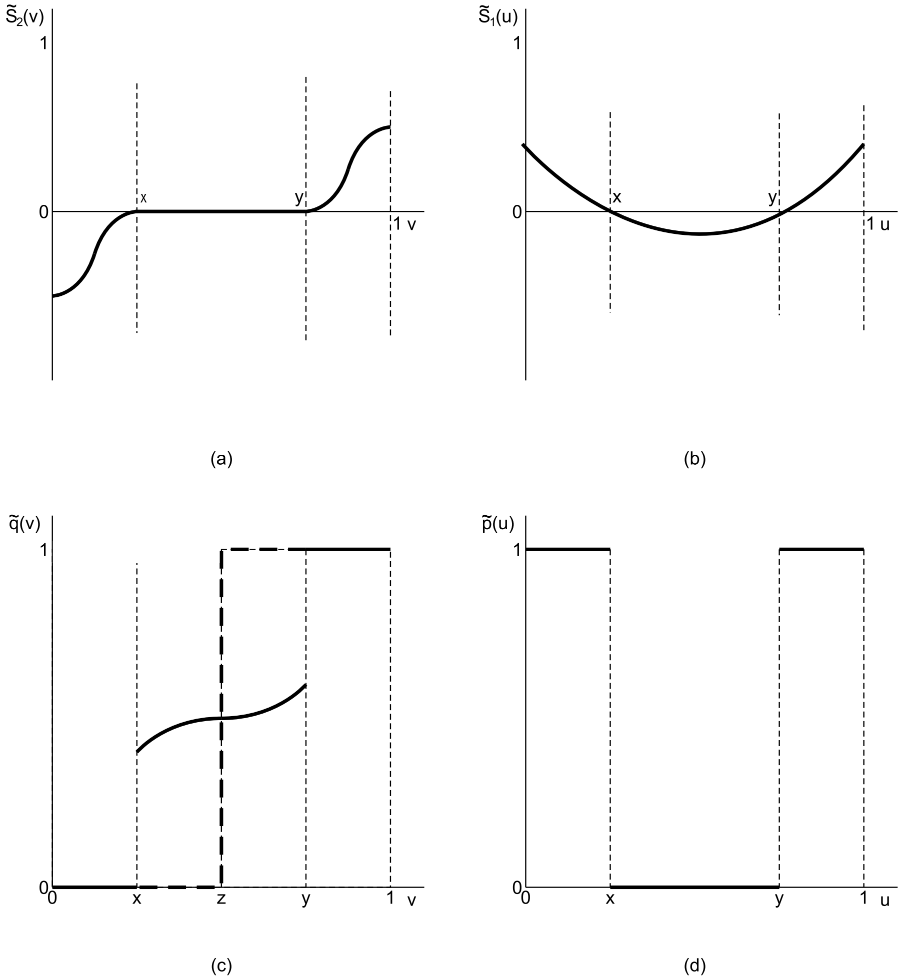


Figure 2: Finding the equilibrium strategies $\widetilde{p}$ and $\widetilde{q}$.

What we have learned about $\widetilde{p}$ tells us something about $\widetilde{S}_1$. It cannot be that $\widetilde{S}_1(u) < 0$ immediately to the left of $x$, because then $\widetilde{p}(u) = 0$ immediately to the left of $x$. Because $\widetilde{S}_1$ is continuous, it follows that $\widetilde{S}_1(x) \geq 0$. For similar reasons $\widetilde{S}_1(y) \geq 0$. Figure 2(c) tells us that $\widetilde{q}(u) = 0$ for every $u$ on the interval $(0, x)$ and that $\widetilde{q}(u) = 1$ on the interval $(y, 1)$. Within these intervals, we can differentiate $\widetilde{S}_1(u)$ with respect to $u$, to obtain that

$$\frac{d}{du}\widetilde{S}_1(u) = -2af(u) + 2(a+b)\widetilde{q}(u)f(u) \quad \text{for all } u \in (0,x) \cup (y,1).$$

Thus $\frac{d}{du}\widetilde{S}_1(u) < 0$ for $u \in (0, x)$ and $\frac{d}{du}\widetilde{S}_1(u) > 0$ for $u \in (y, 1)$. Consequently, $\widetilde{S}_1$ decreases on $[0, x]$ and increases on $[y, 1]$, as indicated in figure 2(b).

Figure 2(b) enables us to tie down $\widetilde{p}$ completely. We already know that $\widetilde{p}(u) = 0$ for $u \in (x, y)$. But now we know that $\widetilde{S}_1(u) > 0$ on $[0, x)$ and $(y, 1]$. Thus, $\widetilde{p}(u) = 1$ on these intervals, as figure 2(d) shows.

Next, use the information about $\widetilde{p}$ and $\widetilde{q}$, together with the fact that $\widetilde{S}_1(x) = \widetilde{S}_1(y) = \widetilde{S}_2(x) = \widetilde{S}_2(y) = 0$ to see that $x$ and $y$ are determined by the equations

$$F(y) = 1 - \frac{2a+b}{b}F(x) \qquad \text{and} \qquad F(y) = \frac{a+b}{2a+b} + \frac{a}{2a+b}F(x)$$

Since $F$ is assumed to have a smooth structure, we can solve these equations to find

$$x = F^{-1}\left(\frac{ab}{(a+b)(4a+b)}\right) \qquad \text{and} \qquad y = F^{-1}\left(\frac{(2a+b)^2 - 2a^2}{(a+b)(4a+b)}\right). \tag{1}$$

So $\widetilde{p}$ is determined uniquely. However, $\widetilde{q}$ is not. For $x \leq v < y$, $\widetilde{q}(v)$ can be chosen freely, subject to the constraints

$$1 - F(y) = \int_x^y q(v)f(v)dv \qquad \text{and} \qquad \widetilde{S}_1(u) \leq 0 \text{ for } x < u < y.$$

These constraints boil down to

$$\begin{aligned} \frac{1}{F(y)-F(x)} \int_x^y q(v)f(v)dv &= \frac{a}{a+b}, \\ \frac{1}{F(y)-F(u)} \int_u^y q(v)f(v)dv &\geq \frac{a}{a+b} \quad \text{for } x < u < y. \end{aligned}$$

Verbally, $\widetilde{q}$ is constrained such that between $x$ and $y$ the average of $\widetilde{q}(v)$ is $\frac{a}{a+b}$, and on any right end of this interval the average of $\widetilde{q}(v)$ is at least $\frac{a}{a+b}$. Although there are many choices for $\widetilde{q}$ that satisfy these constraints, there is a unique admissible Nash equilibrium strategy that does this. A strategy is said to be *admissible* for a player if no other strategy for that player does better against one strategy of the opponent without doing worse against some other strategy of the opponent. This is the strategy with which player 2 folds when his hand is under a certain threshold value $z$ and calls when his hand is above it, such that

$$\int_z^y f(v)dv = \int_y^1 f(v)dv.$$

It follows that this unique value of $z$ is given by

$$z = F^{-1}\left(\frac{b(3a+b)}{(a+b)(4a+b)}\right). \tag{2}$$

This admissible strategy is already indicated in Figure 2(c). Using the strategies $\widetilde{p}$ and $\widetilde{q}$ we computed, we can derive the value of the minipoker game. In Figure 3 all possible hands for player 1 are set out horizontally, together with the action chosen for each hand $u$. For player 2, the hands $v$ and corresponding actions are set out vertically. In each of the ten areas that appear, we know the combination of actions chosen by both players and thus we can give the payoff for each possible combination of hands. By multiplying the area size with this payoff,

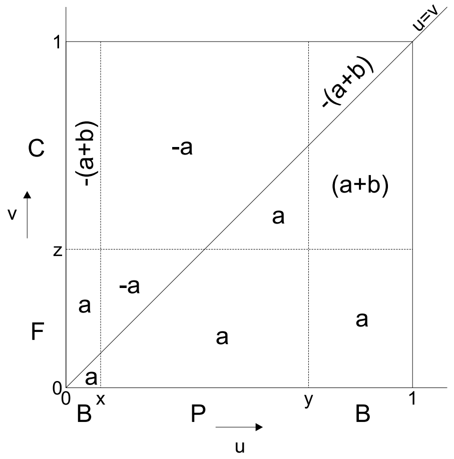


Figure 3: Expected payoff for all $(u, v)$ to player 1 in the Nash equilibrium $(\widetilde{p}, \widetilde{q})$.

and consequently summing over all ten areas, we can compute the value of the game.

$$
\begin{aligned}
v_{a,b} &= a\left(\int_0^y \int_0^u f(v)dvf(u)du + \int_y^1 \int_0^z f(v)dvf(u)du + \int_0^x \int_u^z f(v)dvf(u)du\right) \\
&\quad -a\int_x^y \int_u^1 f(v)dvf(u)du + (a+b)\int_y^1 \int_z^u f(v)dvf(u)du \\
&\quad -(a+b)\left(\int_0^x \int_z^1 f(v)dvf(u)du + \int_y^1 \int_u^1 f(v)dvf(u)du\right) \\
&= 2a\int_x^1 F(u)f(u)du + 2b\int_y^1 F(u)f(u)du - \frac{a(4a+3b)}{4a+b}, \qquad (3)
\end{aligned}
$$

where $x$ and $y$ are as defined in equation (1). The results of the analysis above are summarized in Theorem 3.1.

**Theorem 3.1** *If minipoker is played with ante $a$ and bet size $b$ and the hands $u$ and $v$ of the players both have cumulative distribution $F$ and density $f$ on $[0, 1]$, then the value of the game is given by*

$$v_{a,b} = 2a\int_x^1 F(u)f(u)du + 2b\int_y^1 F(u)f(u)du - \frac{a(4a+3b)}{4a+b},$$

*with* $x = F^{-1}\left(\frac{ab}{(a+b)(4a+b)}\right)$ *and* $y = F^{-1}\left(\frac{(2a+b)^2-2a^2}{(a+b)(4a+b)}\right)$. *In this case, optimal strategies are*

$$Pr\{\textit{bet with hand } u\} = \widetilde{p}(u) = \begin{cases} 1 & \textit{if } Pr\{V \le u\} \le \frac{ab}{(a+b)(4a+b)} \textit{ or } Pr\{V \le u\} > \frac{(2a+b)^2-2a^2}{(a+b)(4a+b)}, \\ 0 & \textit{otherwise,} \end{cases}$$

*for player 1 and*

$$Pr\{\textit{call with hand } v\} = \widetilde{q}(v) = \begin{cases} 0 & \textit{if } Pr\{U \le v\} \le \frac{b(3a+b)}{(a+b)(4a+b)}, \\ 1 & \textit{otherwise,} \end{cases}$$

*for player 2.*

The results for the case of Von Neumann and Morgenstern (1944) where $F$ is the uniform distribution on $[0, 1]$, follow directly from Theorem 3.1.

**Corollary 3.2** *The minipoker game of Von Neumann and Morgenstern (1944), in which F was the uniform distribution, has value*

$$v_{a,b} = \frac{a^2 b}{(a+b)(4a+b)} = ax.$$

*Optimal strategies are given by*

$$Pr\{bet\ with\ hand\ u\} = \widetilde{p}(u) \begin{cases} 1 & if\ u \leq \frac{ab}{(a+b)(4a+b)} \ \ or\ u > \frac{(2a+b)^2 - 2a^2}{(a+b)(4a+b)}, \\ 0 & otherwise, \end{cases}$$

*for player 1 and*

$$Pr\{call\ with\ hand\ v\} = \widetilde{q}(v) = \begin{cases} 0 & if\ v \leq \frac{b(3a+b)}{(a+b)(4a+b)}, \\ 1 & otherwise. \end{cases}$$

*for player 2.*

So, in this simple case, the value of the game is equal to the product of the ante and the value of the hand that indicates player 1's strategic boundary between bluffing and passing. Interesting is the fact that the value is positive in this case. The game is favourable for player 1. To see for what combination of values of the ante and the bet size the game is most favourable for player 1, we fix the ante $a$ and compute the derivative of $v_{a,b}$ with respect to $b$.

$$\frac{d}{db} v_{a,b} = \frac{a^2(2a-b)(2a+b)}{(a+b)^2(4a+b)^2}.$$

This derivative is zero at $b = 2a$. This is the only solution, since both $a$ and $b$ are positive. Since $\frac{d^2}{db^2} v_{a,b} = -\frac{1}{81a} < 0$ for these relative values of the bet size and the ante, we know that the ratio $\frac{b}{a} = 2$ is optimal for player 1. This special case is called pot-limit minipoker, since the maximal bet size (in this case the only possible bet size) equals the total size of the pot. Now we can formulate Proposition 3.3.

**Proposition 3.3** *The pot-limit variant of minipoker with uniform hand distributions is the unfairest variant possible.*

Another thing that is intuitively clear, is easy to recognize now too: for minipoker with uniform distributions the strategies of the players depend only on the ratio $\frac{b}{a}$ of the bet size and the ante and not on the absolute values of $b$ and $a$. If we define the ratio $r$ as $r = \frac{b}{a}$ and substitute this information in the expressions for the boundary values $x$, $y$ and $z$ given in equations (1) and (2), we find that

$$x = \frac{r}{(r+4)(r+1)}, \quad y = \frac{r^2+4r+2}{(r+4)(r+1)} \quad \text{and} \quad z = \frac{r^2+3r}{(r+4)(r+1)}.$$

This fact is displayed in Figure 4, in which we can also see the limits

$$\lim_{r\to\infty} x = 0, \quad \lim_{r\to\infty} y = \lim_{r\to\infty} z = 1, \quad \lim_{r\downarrow 0} x = \lim_{r\downarrow 0} z = 0 \quad \text{and} \quad \lim_{r\downarrow 0} y = \frac{1}{2}.$$

The shapes of these curves for larger values of $r$ is intuitively clear: when betting and calling

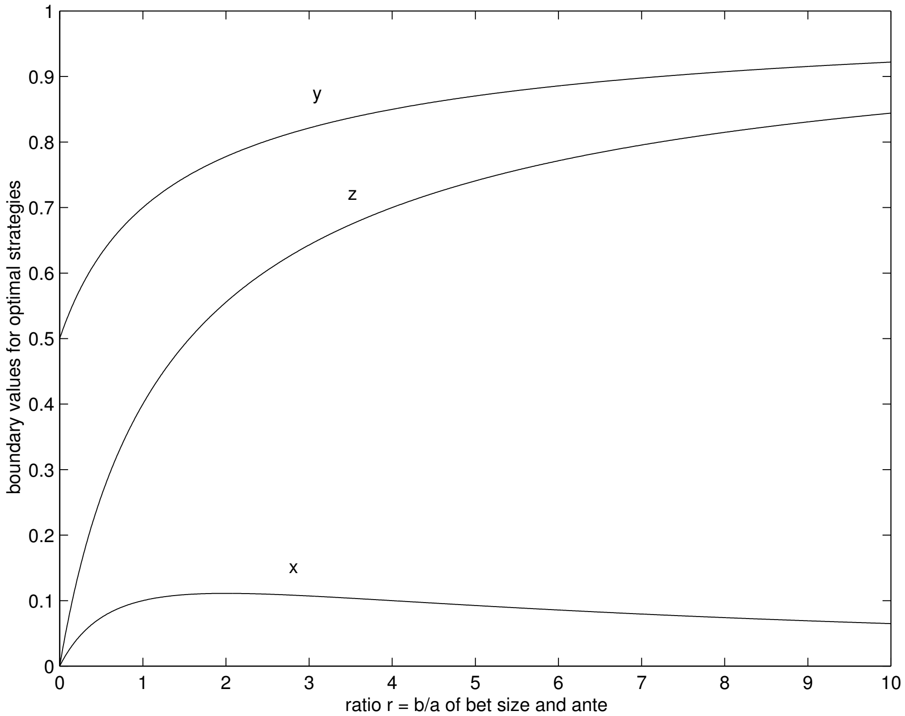


Figure 4: Boundary values for the optimal strategies as a function of the ratio $\frac{b}{a}$ of the bet size and the ante.

becomes relatively expensive, it is wise to do it not too often. As the ratio goes to zero, the number of hands with which player 2 calls increases quickly. Giving up the ante by folding becomes relatively expensive. As a consequence, player 1 only folds with the higher half of the hands, for which the probability that he has the highest hand is larger than $\frac{1}{2}$. Finally, at $r = 2$, the case of pot-limit poker, we see that player 1 has the largest bluffing area.

# 4 A regular deck of cards

In the previous section we derived optimal strategies in the two-person poker game for both players in a general form. These strategies were given in terms of quantiles of the continuous distribution function $F$, the distribution function from which the hands of the players were drawn. In this section we will see what these results imply when the game is played with a regular deck of cards, from which the players draw real poker hands.

### 4.1 Classification of poker hands

Before we start translating strategies, let us first give an overview of the poker hands that can occur. A poker hand is a combination of five cards, drawn from a deck of 52 cards. The deck consists of four suits: hearts ($\heartsuit$), clubs ($\clubsuit$), diamonds ($\diamondsuit$) and spades ($\spadesuit$). All suits are equally valuable, while the 13 cards of each suit have, ranked in decreasing order, the values $A$(ce), $K$(ing), $Q$(ueen), $J$(ack), 10, 9, ..., 2. All hands belong to one of the ten *classes* that are defined in decreasing order of value in Table 1. The order of hands within a class is determined

| | Class | Description | Example |
|---|---|---|---|
| *RF* | Royal Flush | five consecutive cards of one suit, starting with an ace | $(\clubsuit A, \clubsuit K, \clubsuit Q, \clubsuit J, \clubsuit 10)$ |
| *SF* | Straight Flush | five consecutive cards of the same suit (an ace can have the value 1) | $(\spadesuit 5, \spadesuit 4, \spadesuit 3, \spadesuit 2, \spadesuit A)$ |
| *4K* | Four of a Kind | four cards with equal values | $(\diamondsuit 4, \clubsuit 4, \heartsuit 4, \spadesuit 4, \diamondsuit Q)$ |
| *FH* | Full House | a triplet of cards with the same values, together with a pair with equal values | $(\spadesuit 5, \clubsuit 5, \diamondsuit 5, \diamondsuit 10, \heartsuit 10)$ |
| *F* | Flush | five cards of the same suit | $(\clubsuit K, \clubsuit J, \clubsuit 9, \clubsuit 3, \clubsuit 2)$ |
| *S* | Straight | five consecutive cards | $(\heartsuit K, \spadesuit Q, \heartsuit J, \clubsuit 10, \diamondsuit 9)$ |
| *3K* | Three of a Kind | three cards with the same value | $(\clubsuit Q, \heartsuit Q, \spadesuit Q, \diamondsuit J, \heartsuit 6)$ |
| *2P* | Two pair | two pairs with the same values within each pair | $(\spadesuit A, \heartsuit A, \diamondsuit 8, \spadesuit 8, \clubsuit 3)$ |
| *1P* | One pair | one pair of cards with equal values | $(\heartsuit 9, \diamondsuit 9, \clubsuit K, \diamondsuit 10, \diamondsuit 4)$ |
| *HC* | High Card | any combination of cards that does not fit in any of the classes above | $(\heartsuit K, \diamondsuit J, \diamondsuit 9, \clubsuit 4, \spadesuit 2)$ |

Table 1: Classification of poker hands

by comparing the cards of the hands separately, starting with the most important card of a hand. The importance of the card within a hand depends on the class to which the hand belongs. In Table 1 the card order in the example hands is such that the most important cards are put in front.

The total number of different hands of five cards that can be drawn from a single deck of 52 cards is $\binom{52}{5} = 2,598,960$. The number of hands in each class and the probability of receiving a hand from this class is given in Table 2 for all ten classes. The decreasing probabilities are the reason that the order of the classes is as it is. If we pay attention to the order of the hands within the ten classes, then we obtain $7,462$ ordered *subclasses*. Within each subclass, all hands really are equal. In Figure 5 we give the frequencies with which hands of a certain subclass appear. The small bar with high frequencies around subclass number $7,200$ corresponds to the Straights, while the somewhat wider block with frequencies of 24 corresponds to Full House. Figure 6 gives the continuous approximation of the cumulative distribution of the poker hands,

| Class | Number | Prob.(%) |
|---|---|---|
| *RF* | 4 | 0.000 |
| *SF* | 36 | 0.001 |
| *4K* | 624 | 0.024 |
| *FH* | 3,744 | 0.144 |
| *F* | 5,108 | 0.197 |
| *S* | 10,200 | 0.392 |
| *3K* | 54,912 | 2.113 |
| *2P* | 123,552 | 4.754 |
| *1P* | 1,098,240 | 42.257 |
| *HC* | 1,302,540 | 50.118 |
| Total | 2,598,960 | 100.000 |

Table 2: Numbers and probabilities for all classes of poker hands

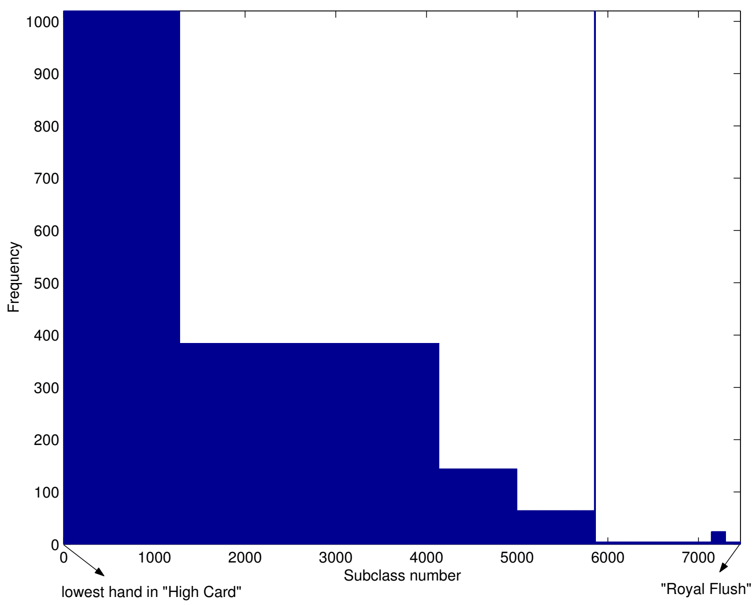


Figure 5: Frequencies of appearance of subclasses of poker hands in a single deck of 52 cards.

where both hand numbers and frequencies are normalized.

### 4.2 From a continuous to a discrete distribution

All results we presented so far were derived using continuous hand distributions. Now we want to take these results from the continuous situation into the discrete real world, where the hands are drawn from a deck of 52 cards, with or without replacement. An intuitive way to approximate optimal strategies in the discrete game is the following. If the 5-card hand of a player ranks $n$ (from the bottom) out of 2,598,960, we treat his hand as if he were dealt $\frac{n}{2{,}598{,}960}$ in the

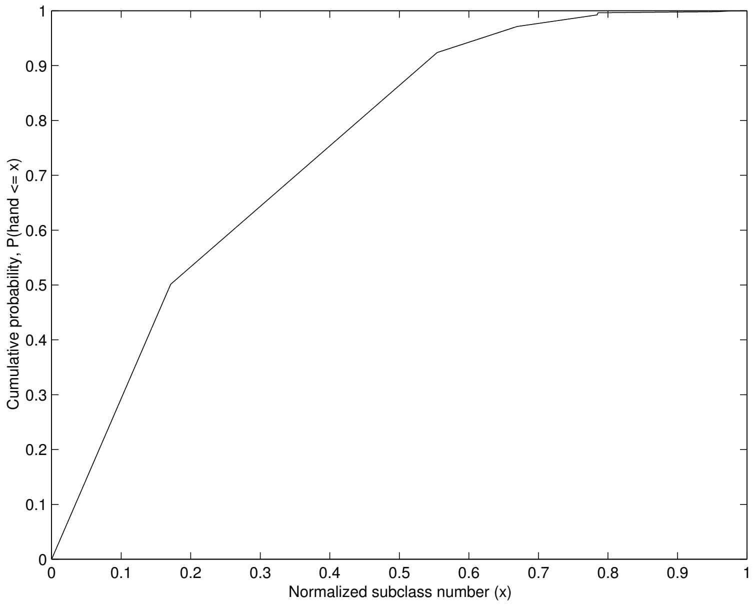


Figure 6: Continuous approximation of the cumulative distribution of the 7,462 subclasses of poker hands in a single deck of 52 cards.

continuous game.

As Cutler (1975) suggests, there are at least three objections to this approximation. First of all, the optimal strategies for the discrete case may differ considerably from the ones derived for the continuous case. However, according to Von Neumann and Morgenstern (1944, p. 209), the maximal loss that can be incurred by playing the "continuous" strategy is not large. More precisely, the difference is only of the order $\frac{1}{2{,}598{,}960}$. Second, some different hands have an equal value, as the ordering (partly) disregards suits. This fact is taken care of by using the general distribution $F$ in our derivation. Even if certain hands occur with higher probability than others, our results still apply. Finally, the hands are dealt from one deck without replacement. That is, the hand one player holds affects what the other may hold. As a result, increasing the rank of a hand does not necessarily increase its value. Consider the following example. If you hold a straight flush to the five, your opponent may hold 31 higher straight flushes or three equal ones. However, if you hold four aces and a six, your opponent may only beat you with 27 different straight flushes. We will not take into account this last remark and focus on the case where minipoker is played with a separate deck of cards for each player. Or equivalently, it could be interpreted as the game in which the players' hands are drawn from a regular deck of 52 cards with replacement. We will give an approximation for optimal play for this game in section 4.3.

### 4.3 Optimal play

In this section we will tell what the optimal minipoker strategies for both players mean in terms of real poker hands. We consider the case where hands are drawn from a regular deck of cards with replacement. Unless stated otherwise explicitly, the results in this section apply to the case where the ante and the bet size are equal, i.e. $r = \frac{b}{a} = 1$. Recall from Theorem 3.1 that if the players are dealt the hands $u$ and $v$, the optimal strategy for player 1, stated in terms of probabilities, for this ratio is

$$Pr\{\text{bet with hand } u\} = \widetilde{p}(u) = \begin{cases} 1 & \text{if } Pr\{V \leq u\} \leq \frac{1}{10} \text{ or } Pr\{V \leq u\} > \frac{7}{10}, \\ 0 & \text{otherwise,} \end{cases}$$

and that it is optimal for player 2 to play

$$Pr\{\text{call with hand } v\} = \widetilde{q}(v) = \begin{cases} 0 & \text{if } Pr\{U \leq v\} \leq \frac{2}{5}, \\ 1 & \text{otherwise.} \end{cases}$$

Using the information that is displayed in Figure 6, we can translate these probabilities to the probabilities of poker hands. We find that the nearly optimal strategy for player 1 is

$$Pr\{\text{bet with hand } u\} = \begin{cases} 1 & \text{if } u \leq (Q, 7, 5, 4, 3), \\ 0 & \text{if } (Q, 7, 6, 3, 2) \leq u \leq (8, 8, 9, 5, 4), \\ 1 & \text{if } u \geq (8, 8, 9, 6, 2), \end{cases}$$

and that it is approximately optimal for player 2 to play

$$Pr\{\text{call with hand } v\} = \begin{cases} 0 & \text{if } v \leq (A, Q, 8, 6, 2), \\ 1 & \text{if } v \geq (A, Q, 8, 6, 3). \end{cases}$$

To be precise, for player 1 the hands are selected such that $(Q, 7, 5, 4, 3)$ is the highest hand for which $Pr\{V \leq u\} \leq \frac{1}{10}$ and $(Q, 7, 6, 3, 2)$ is the lowest hand for which $Pr\{V \leq u\} > \frac{1}{10}$. To indicate the dependency of the strategies on the ratio of bet size and ante, that was shown for the uniform case in Figure 4, Table 3 gives the boundary hands for some other relative values of $a$ and $b$. In this table, $x^-$ is the highest hand below the boundary $x$. The definitions for $y^-$ and $z^-$ are analogous. The case $r = 1$ is included to compare with the results above. In Table 3 we can clearly see that, with a relatively high cost of betting and calling, optimal play prescribes betting and calling only for a small number of hands.

## 5 Relative skill

In this section we will study the game in which the hands $u$ and $v$ for player 1 and 2 are drawn independently from a uniform distribution on $[0, 1]$. We focus on the case with equal ante and bet size again and normalize to $a = b = 1$. We will follow the analysis of skill analysis that was proposed by Borm and Van der Genugten (2001). We will give a short description of this *relative skill measure* for two-player games in the next section.

| $r = \frac{b}{a}$ | $x^-$ (player 1's upper bound for bluffing) | $y^-$ (player 1's lower bound for betting) | $z^-$ (player 2's lower bound for calling) |
|---|---|---|---|
| 1 | $(Q, 7, 5, 4, 3)$ | $(8, 8, 9, 5, 4)$ | $(A, Q, 8, 6, 2)$ |
| 2 | $(Q, 9, 5, 4, 2)$ | $(10, 10, Q, J, 2)$ | $(3, 3, K, J, 2)$ |
| 3 | $(Q, 8, 7, 4, 3)$ | $(J, J, A, 9, 3)$ | $(6, 6, J, 10, 3)$ |
| 5 | $(J, 10, 9, 6, 5)$ | $(K, K, 10, 9, 3)$ | $(9, 9, J, 10, 6)$ |
| 10 | $(J, 9, 6, 5, 4)$ | $(A, A, K, J, 10)$ | $(Q, Q, K, 4, 2)$ |
| 100 | $(9, 7, 5, 3, 2)$ | $(K, K, K, Q, 2)$ | $(7, 7, 7, K, 4)$ |

Table 3: Boundary values of the optimal strategies for both players for various ratios $r = \frac{b}{a}$.

### 5.1 The relative skill measure

For any game we distinguish three types of players: beginners, optimal players and fictive players. Beginners have just learned the rules of the game and play a naive strategy, while the optimal players play a minimax strategy against their opponent. The fictive players play optimal too, but they have more information; they know the complete outcome of the *external chance moves* before they have to decide what action to take. In minipoker this means that a player knows what hand the opponent holds. In a two-person game these three types can participate in the game in both player roles. Skill is defined as the relative influence of the players on the outcome of the game. To measure this, one computes two effects in the game, the *learning effect (LE)* and the *random effect (RE)*. The learning effect is defined as the difference in expected payoff between an optimal player and a beginner, while the random effect is the difference between the expected payoff of a fictive player and the expected payoff of an optimal player. To compute expected payoffs in the strategic environment of the poker game, we also need to know the strategy of the opponent. All three player types are evaluated against the same type of opponent, namely one that plays the minimax strategy. To find the expected payoffs of a specific player type in a two-person game, we take the average over the two player roles. When we refer to strategies we will use the subscript 0 to indicate that it is a strategy that is used by a beginner, while a strategy with the subscript $f$ corresponds to a fictive player. If we introduce the notation $U_i(s_1, s_2)$ for the expected payoff to player $i$ when player 1 plays strategy $s_1$ and player 2 uses strategy $s_2$, then we can write down the expressions for the learning effect, the random effect and the relative skill measure $RS$.

$$LE = \frac{1}{2}\left(U_1(\widetilde{p}, \widetilde{q}) + U_2(\widetilde{p}, \widetilde{q}) - U_1(p_0, \widetilde{q}) - U_2(\widetilde{p}, q_0)\right) \tag{4}$$

$$RE = \frac{1}{2}\left(U_1(p_f, \widetilde{q}) + U_2(\widetilde{p}, q_f) - U_1(\widetilde{p}, \widetilde{q}) - U_2(\widetilde{p}, \widetilde{q})\right) \tag{5}$$

$$RS = \frac{LE}{LE + RE} = \frac{-U_1(p_0, \widetilde{q}) - U_2(\widetilde{p}, q_0)}{U_1(p_f, \widetilde{q}) + U_2(\widetilde{p}, q_f) - U_1(p_0, \widetilde{q}) - U_2(\widetilde{p}, q_0)} \tag{6}$$

In this expression, $p$'s correspond to strategies of player 1 and $q$'s denote strategies of player 2. It is easy to see from equations (4)-(6) that $0 \le RS \le 1$. The limit cases $RS = 0$ and $RS = 1$ correspond to *pure games of chance* and *pure games of skill* respectively.

In section 5.2 we will present our assumptions on the behaviour of beginners in minipoker as well as the resulting payoffs for these players, while section 5.3 contains the derivation of the strategies that fictive players use and the computation of their corresponding payoffs.

## 5.2 Beginners

What will be the strategies of players who play this game for the first time, just after the rules are explained to them? Perhaps they heard about the famous video poker variant "Jacks or Better". In this game, as the name suggests, only hands with a pair of Jacks, Queens, Kings or Aces (and all hands from higher classes) have value for the player. As a result naive players may be betting or calling with exactly these hands. Even if they do not know this game, this border seems to be a reasonable one. After all, poker players tend to like hands that look fancy; any hand with at least a pair of images surely satisfies this condition of prettiness.

What does this reasoning mean for the strategies of the beginners? Player 1 bets only if his hand is at least $(J, J, 4, 3, 2)$. For each player the total probability of receiving a hand up to $(J, J, 4, 3, 2)$ is $\frac{1189}{1498} \approx 0.7937$. So we can formulate the strategy for player 1 as a beginner as

$$p_0(u) = \begin{cases} 0 & \text{if } 0 \le u \le 0.7937 \\ 1 & \text{if } 0.7937 \le u \le 1, \end{cases}$$

while the beginner's strategy for player 2 can be formulated as

$$q_0(v) = \begin{cases} 0 & \text{if } 0 \le v \le 0.7937 \\ 1 & \text{if } 0.7937 \le v \le 1. \end{cases}$$

Both strategies are displayed graphically in Figure 7. In Figure 8 the expected payoff to player 1

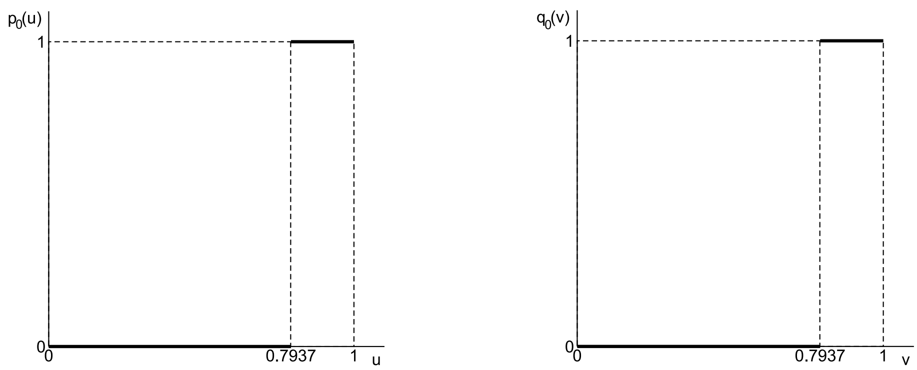


Figure 7: The strategies $p_0$ and $q_0$ for beginning player 1 and player 2 respectively.

is given for all hand distributions $(u, v)$, assuming that player 1 uses strategy $p_0$ and player two

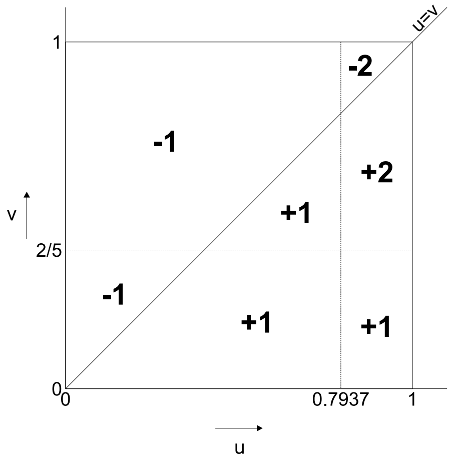


Figure 8: Expected payoff for all $(u, v)$ to player 1 if he plays as a beginner against the equilibrium strategy $\widetilde{q}$ of player 2.

plays the strategy $\widetilde{q}$, that is given in Corollary 3.2. Using this figure, one can sum over a number of simple integrals to find that the expected payoff to player 1 as a beginner is

$$U_1(p_0, \widetilde{q}) = \frac{310}{3817} \approx 0.0812.$$

We do the same computation for player 2, using Figure 9 in which the expected payoffs to player 1 for the strategy combination $(\widetilde{p}, q_0)$ are shown and find that

$$U_2(\widetilde{p}, q_0) = -\frac{265}{2254} \approx 0.1176.$$

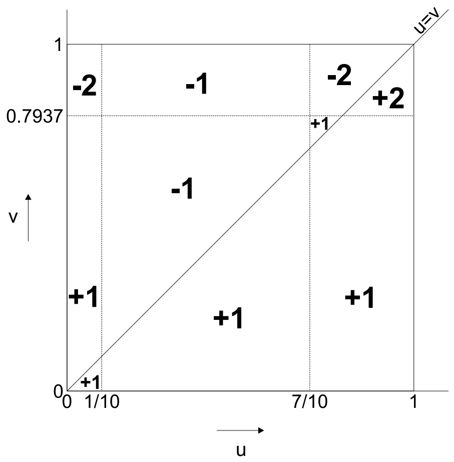


Figure 9: Expected payoff for all $(u, v)$ to player 1 if player 2 plays as a beginner against equilibrium strategy $\widetilde{p}$ of player 1 that was given in Corollary 3.2.

### 5.3 Fictive players

In the current section we will compute the expected payoffs of fictive players in minipoker. Fictive players have more information than normal players. They know the outcome of the chance move in the game and they can use this information in their strategies. For minipoker, this means that the fictive player can base his actions on his own hand, but also on the hand of his opponent. Given the fact that he plays against a player who uses the minimax strategy, he can decide what will be his best action for any hand combination $(u, v)$.

Figure 10 shows the payoff to player 1 for each hand combination if player 1 plays as a fictive player against player 2's equilibrium strategy $\widetilde{q}$. The payoffs in the figure are such that player 1 takes the optimal action for each pair of hands $(u, v)$. For example, in the area above the line $v = \frac{2}{5}$ and above the line $u = v$, player 1 knows that player 2 will always call. Since player 1 has the lower card, he had better pass. This leads to the expected payoff of $-1$ that the figure displays for this area. The expected gains of player 1 can now be computed with help of Figure 10 and are equal to

$$U_1(p_f, \widetilde{q}) = \frac{17}{50}.$$

Figure 11 shows the payoff to player 1 for each card combination if player 2 plays as a fictive

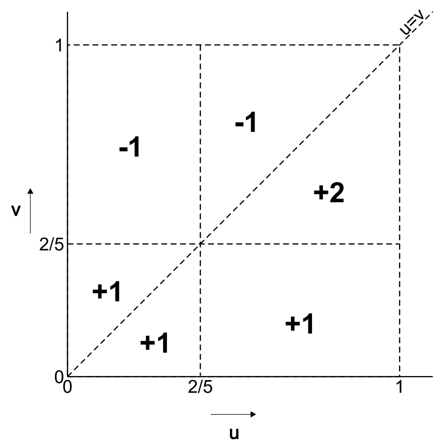


Figure 10: Expected payoff for all $(u, v)$ to fictive player 1 if player 2 uses the equilibrium strategy $\widetilde{q}$.

player against player 1's equilibrium strategy $\widetilde{p}$. The expected gains for player 2 as a fictive player can now be computed with help of this figure and are equal to

$$U_2(\widetilde{p}, q_f) = \frac{7}{50}.$$

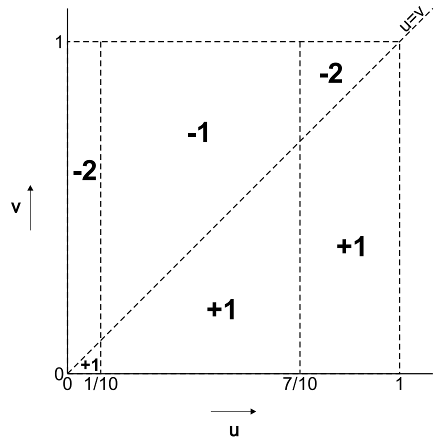


Figure 11: Expected payoff for all $(u, v)$ to player 1, using his equilibrium strategy $\widetilde{p}$, if he faces a fictive player 2.

| | Player 1 | Player 2 | Game |
|---|---|---|---|
| Beginner | 0.0812 | -0.1176 | -0.0182 |
| Optimal | 0.1000 | -0.1000 | 0.0000 |
| Fictive | 0.3400 | 0.1400 | 0.2400 |
| *LE* | 0.0188 | 0.0176 | 0.0182 |
| *RE* | 0.2400 | 0.2400 | 0.2400 |
| *RS* | 0.0726 | 0.0682 | 0.0704 |

Table 4: Results of the skill analysis.

### 5.4 Results of the skill analysis

In the previous two sections we computed the expected payoffs of the beginners and the fictive players. For our two-person game we have enough information to compute the learning effect, the random effect and the skill level according the formulas (4)-(6). The resulting numbers are given in Table 4. In this table we also give the results of the skill analysis for each player separately. The table illustrates that the game does not contain many skill elements for either of the players. The relatively large random effects given in Table 4 tell us that the dealing has an influence on the possibilities of both players that certainly is not negligible. This is an intuitive result for this poker game with a very small range of strategic options for the players. Most poker games have a more complex decision tree, which can for example include a number of raises, multiple bet sizes and a draw. We expect that such an increase in the complexity of the decision tree will also heighten the skill level of the game.

Although this skill level 0.0704 is close to zero, it is still relatively large, if compared to the skill levels that Van der Genugten, Das and Borm (2001) found for games that intuitively classify as games of chance, such as Golden Ten (0.012) and Roulette (0.0004). There is some skill involved in poker, even in this simple variant, as our intuition already suggested.

## Acknowledgement

The authors thank Mark Voorneveld for his valuable comments upon an early version of this note.


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